On a New Method of Determining the General Perturbations of the Minor Planets.

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In finding the general perturbations of the minor planets the special difficulty arises from the large eccentricity and inclination of these bodies. The methods used in case of the major planets fail when applied to the minor, on account of want of convergence in the series. Astronomers were content, therefore, for a long time, with computing the special perturbations of these bodies from epoch to epoch. Hansen finally succeeded in effecting a solution of the problem, and his work entitled, *Auseinandersetzung einer Zweckmässigen Methode zur Berechnung der Absoluten Stöungen der Kleiner Planeten*, contains all the formulæ necessary in the cases thus far occurring.

Instead of determining the perturbations of the coördinates, rectangular or polar, or of finding the variations of the elements, as had been done by his predecessors, Hansen, in his mode of treatment, regards the elements as constant, and finds what we may term the perturbation of the time. Thus, in place of the time, he uses a function of the time, which he designates by z; so that if g_0 is the mean anomaly at the epoch, we have the mean anomaly at any time, in the disturbed orbit, given by $g_0 + n_0 z$, n_0 being the mean daily motion, and being one of the constants. If there were no perturbations we should have $g_0 + n_0 t$, t being the time elapsed since the epoch.

In effecting his solution of the problem, Hansen does not attempt to give general and complete analytical expressions of the series. Instead, he, at the start, converts the coefficients into numbers, and multiplies the series together, two and two, by the methods of trigonometry. Thus, although we find, finally, the perturbations as functions of the time, that is, have the general perturbations, yet, in applying the method to different bodies, we must find the values of all the quantities involved for the particular case under consideration. It would be a great advance if we had at hand complete analytical expressions, of sufficient convergence, as is the case with the larger planets.

Besides the method of multiplying series together by the methods of trigonometry, which Hansen calls "Mechanical Multiplication"—a method he was the first to employ—he also adopts different angles with which to express his arguments. Thus at the outstart he uses the eccentric anomaly for both bodies. When he has computed the powers of the reciprocal of the distance between the disturbed and disturbing bodies, he transforms from eccentric to mean anomaly in case of the disturbing body. And then, when he has expressions for the perturbing function and the forces, he makes another transformation so as to be able to effect the integrations.

The transformations must be done with great care, and require a large measure of time. In addition to the tedium arising from extended operations of this kind (which must be generally done in duplicate to insure accuracy), many of the processes in various stages of the work are not easily grasped, and certainty is often only secured by performing the numerical calculations. Thus, then, although the method has been published for a long time, it has been applied only in a very limited number of cases. Watson, in the Preface to his *Theoretical Astronomy*, says: "The refined and difficult analysis and the laborious calculations involved were such that, even after Hansen's methods were made known, astronomers still adhered to the method of special perturbations by the variation of constants as developed by Lagrange."

Hansen seems himself to have felt the force of these drawbacks on his method, as in a posthumous memoir devoted to the larger planets he abandons his peculiar method of treatment and uses that of Lagrange.

As far as the minor planets are concerned, there is no doubt that Hansen's method, as left by him, is too long and difficult to be practicable.

What we need now is some mode of determining general perturbations that is easily applied and sufficiently short to attract the efforts of a larger number of competent computers. Only in this way can the constantly growing material be utilized. The new method of treatment will now be given as briefly as possible.

If Δ be the distance between the disturbed and disturbing bodies, Hansen has the equation

$$\left(\frac{a}{\Delta}\right)^{n} = \left\{ C - q \cos(\varepsilon' - Q) \right\}^{-\frac{n}{2}} \left\{ 1 - q_1 (\cos \varepsilon' + Q) \right\}^{-\frac{n}{2}}$$

for finding $\left(\frac{a}{\Delta}\right)$, $\left(\frac{a}{\Delta}\right)^3$, etc.

Instead of the two factors of the second member, I have used a transformation of them given by Hill, and have

$$\left(\frac{a}{\Delta}\right)^{n} = N^{n} \left(1 + a^2 - 2a \cos(\varepsilon' - \theta)\right)^{-\frac{n}{2}} \left(1 + b^2 - 2\, b \cos(\varepsilon' + Q)\right)^{-\frac{n}{2}},$$

where $\left(1 + a^2 - 2a \cos(\varepsilon' - \theta)\right)^{-\frac{n}{2}} = \Big[\tfrac{1}{2} b_{\frac{n}{2}}^{(0)} + b_{\frac{n}{2}}^{(1)} \cos(\varepsilon' - Q)$

$+ b_{\frac{n}{2}}^{(2)} \cos 2(\varepsilon' - Q) + \text{etc.}\Big]$,

and similarly for $\left(1 + b^2 - 2b \cos(\varepsilon' + Q)\right)^{-\frac{n}{2}}$.

The coefficients of both these factors are the La Place coefficients, and their values have been tabulated. Thus the part of the work relating to the determination of expressions for $\left(\frac{a}{\Delta}\right)$, $\left(\frac{a}{\Delta}\right)^3$, etc., is rendered comparatively short and simple.

In finding Δ^2 in terms of the radii vectores of the two bodies and of the cosine of the angle between these radii-vectores, the true anomaly of both bodies is introduced. In the analysis we use the equivalent functions of the eccentric anomaly for those of the true anomaly, and then, when making the numerical computations, we cause the eccentric anomaly of the disturbed body to disappear. This is accomplished by dividing the circumference into a certain number of equal parts relative to the mean anomaly, and employing for the eccentric anomaly its numerical values corresponding to the various values of the mean anomaly.

Having found the expressions of $\left(\frac{a}{\Delta}\right)$, $\left(\frac{a}{\Delta}\right)^3$, etc., in series, in which the angles are the mean anomaly of the disturbed and the eccentric anomaly of the disturbing body, the series are changed at once into others in which both angles are mean anomalies. To effect this transformation there is need of functions called the J functions; and a chapter is given in which the expressions for these functions are found in a form convenient for application.

When we have the powers of the reciprocal of the distance between the disturbed and disturbing bodies, we next find the term expressing the effect of the action of the disturbing body on the sun. This is effected without difficulty.

The expressions for the perturbing function and the perturbing forces can now be formed. Instead of using the force involving the true anomaly, the transformation of this, in which the mean anomaly appears instead of the true, has been used. This is the method given by Hansen in his posthumous memoir, in which he has abandoned some of his former notions. The disturbing forces employed are those in the direction of the disturbed radius-vector, in the direction perpendicular to this radius-vector, and in the direction perpendicular to the plane of the orbit. The forces in these three directions have been deduced from those in the direction of the three rectangular axes. The force $a\frac{d\Omega}{dg}$ is found at once from the perturbing function by differentiating with respect to the mean anomaly, g.

To find the other two forces symbolized by $a\, r. \frac{d\Omega}{dr}$, and $a^2\frac{d\Omega}{dz}$, z being the coördinate perpendicular to the plane of the orbit, it is necessary to multiply a number of series together, two and two, by the formulæ of plane trigonometry.

Having the values of the forces, we next find the value of a function W obtained by the integration of the expression

$$\frac{dW}{n.\, dt} = A.\, a\frac{d\Omega}{dg} + B.\, a\, r\frac{d\Omega}{dr},$$

A and B being two factors easily determined. W being known, the function $\overline{W}$ is next found by simple mechanical processes, and the perturbations of z and of the radius-vector are found at once by the equations

$$n.\ \delta z = n \int . \overline{W}.\ dt$$

$$v = -\tfrac{1}{2}\, u \int \frac{d\overline{W}}{d\gamma}.\ dt,$$

γ being a particular form for g.

The symbol δ designates the perturbation of the quantity to which it is prefixed.

The perturbation of the latitude is found by integrating the equation

$$\frac{d. \dfrac{u}{cos i}}{n.dt} = C.\ a^2\ \frac{d\Omega}{dz},$$

where C is a factor found in the same manner that A and B were.

To find $n.\delta z$, or the variation of the mean anomaly, two integrations are necessary; in finding the perturbation of the radius-vector, and of the latitude, one integration is needed for each.

The arbitrary constants introduced by these integrations are so determined that the perturbations become zero for the epoch of the elements.

In making an application of his formulæ, Hansen selected the planet Egeria, whose eccentricity is comparatively small, the angle of eccentricity being less than five degrees. In making use of the formulæ given in the method here presented the eccentricity is considerably larger. The convergence of the series is, however, all that can be desired. In computing the perturbations of those of the minor planets whose eccentricities and inclinations are quite large, it may be necessary to divide the circumference into a larger number of parts. In exceptional cases, such as for Pallas, it may be necessary to divide the circumference into thirty-two parts. In case of the applications made of the present method, sixteen divisions have been used: this is the number employed by Hansen for Egeria.

When a larger number than sixteen is used, the calculation of the values of $\left(\frac{a}{\Delta}\right)$, $\left(\frac{a}{\Delta}\right)^3$, etc., is longer; the process is, however, the same in every case.

After the perturbations have been found it is necessary to have them in convenient form for the computation of ephemerides, and there has, hence, been added the method employed for doing this.

The writer has endeavored to present the whole theory in a manner easily comprehended by those having a respectable mathematical education, and in a compass such that the computations can be performed within a reasonably short time. The endeavor throughout has been to use convenient methods, not to devise new ones.